stabilizing controllers always exist for the matched uncertain linear systems that contain either norm- or entry-bounded uncertainty matrices. The proposed robust stabilizing control law can be determined easily from the symmetric positive-definite solution of the augmented Riccati equation. In addition, the proposed approach is flexible in the sense that some adjustable parameters (such as ϵ, γ, and h, etc.) have been introduced in the derivations to achieve the stabilization of matched uncertain linear systems. Moreover, the proposed method can be applied to unstable and/or nonminimum phase-matched uncertain linear multivariable systems.

Acknowledgments

This work was supported in part by the U.S. Army Research Office under Contract DAAL-03-87-K0001 and by the NASA Johnson Space Center under Grants NAG 9-380 and NAG 9-385. The authors wish to express their gratitude to the reviewers for their valuable comments and suggestions.

References

[1]Leitmann, G., "Guaranteed Asymptotic Stability for Some Linear Systems with Bounded Uncertainties," *Journal of Dynamic Systems, Measurement, and Control*, Vol. 101, No. 3, 1979, pp. 212–216.

[2]Thorp, J. S., and Barmish, B. R., "On Guaranteed Stability of Uncertain Systems via Linear Control," *Journal of Optimization Theory and Applications*, Vol. 35, No. 4, 1981, pp. 559–579.

[3]Barmish, B. R., Corless, M., and Leitmann, G., "A New Class of Stabilizing Controllers for Uncertain Dynamic Systems," *SIAM Journal of Control and Optimization*, Vol. 21, No. 2, 1983, pp. 246–255.

[4]Schmitendorf, W. E., and Barmish, B. R., "Robust Asymptotic Tracking for Linear Systems with Unknown Parameters," *Automatica*, Vol. 22, No. 3, 1986, pp. 355–359.

[5]Jabbari, F., and Schmitendorf, W. E., "A Non-Iterative Method for Design of Linear Robust Controllers," *Proceedings of 28th IEEE Conference on Decision and Control*, IEEE, New York, Dec. 1989, pp. 1690–1692.

[6]Schmitendorf, W. E., "A Design Methodology for Robust Stabilizing Controllers," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 3, 1987, pp. 250–254.

[7]Petersen, I. R., and Hollot, C. V., "A Riccati Equation Approach to the Stabilization of Uncertain Linear Systems," *Automatica*, Vol. 22, No. 4, 1986, pp. 397–411.

[8]Schmitendorf, W. E., "Designing Stabilizing Controllers for Uncertain Systems Using the Riccati Equation Approach," *IEEE Transactions on Automatic Control*, Vol. AC-33, No. 4, 1988, pp. 376–379.

[9]Skelton, R. E., *Dynamic Systems Control*, Wiley, New York, 1988.

[10]Anderson, B. D. O., and Moore, J. B., *Linear Optimal Control*, Prentice-Hall, Englewood Cliffs, NJ, 1990.

Analysis of a Rotationally Accelerated Beam with Finite Tip Mass and Hub

K. Moesslacher Jr.,* J. C. Bruch Jr.,†
and
T. P. Mitchell†
University of California at Santa Barbara,
Santa Barbara, California 93106

Nomenclature

A = cross-sectional area of the beam
a = radius of the hub
b = beam thickness
c = half the length of the tip mass
d = tip mass diameter
E = Young's modulus of elasticity
h = beam height
I = area moment of inertia of beam cross section
I_b = mass moment of inertia of the beam about $x = 0$
I_c = mass moment of inertia of the extended tip mass about its center of gravity
I_h = mass moment of inertia of the hub about center
I_t = mass moment of inertia of tip mass about $x = L$
L = length of the beam
L_h = hub length
L_L = beam length
L_t = tip mass length
M_b = beam mass
M_t = tip mass
P = period
r = radius of tip mass
$T(t)$ = torque applied to hub
T_o = torque magnitude
t = time
$v(x,t)$ = transverse deflection of the beam
x = coordinate along the undeflected beam
θ = hub angle of rotation
ρ = mass density of the beam
ρ_h = hub density
ρ_t = tip mass density
$\tilde{\phi}(x)$ = initial displacement distribution of the beam
$\tilde{\psi}(x)$ = initial velocity distribution of the beam

Received Oct. 5, 1989; revision received May 16, 1990; accepted for publication May 16, 1990.

*Presently Engineer Scientist, McDonnell Douglas Space Systems Company, 5301 Bolsa Avenue, Huntington Beach, CA 92647.

†Professor, Department of Mechanical and Environmental Engineering.

Introduction

THE influence of element flexibility on the motion of a system is the subject of considerable current research.[1-13] In this Note, a rotationally accelerated Bernoulli beam is studied. Analytical expressions for beam tip displacement, hub rotation angle, and beam flexure are obtained. A parameter analysis of natural frequencies is presented.

Mathematical Model

The system shown in Fig. 1 consists of a slender flexible horizontal beam with a rectangular cross section that is attached to a rotating cylindrical rigid body supported in a cantilever fashion. A rigid-body tip is attached to the free end of the beam. The beam rotates in a horizontal plane generating in-plane bending. The nondimensional form of the governing equations neglecting terms in the square of the angular velocity are

Field equation:

$$\ddot{\eta} + \eta'''' + (\xi + \delta)\ddot{\theta} = 0, \qquad \xi \in (0,1) \tag{1}$$

Boundary conditions:

$$\eta(0,\tau) = 0, \qquad \eta'(0,\tau) = 0$$

$$\eta''(1,\tau) = -(\lambda/\sigma)[\ddot{\eta}(1,\tau) + (\delta + 1)\ddot{\theta}] - \zeta[\dot{\eta}''(1,\tau) + \ddot{\theta}]$$

$$\eta'''(1,\tau) = (\lambda/\sigma)[\dot{\eta}''(1,\tau) + \ddot{\theta}] + (1/\sigma)[\ddot{\eta}(1,\tau) + (\delta + 1)\ddot{\theta}] \tag{2}$$

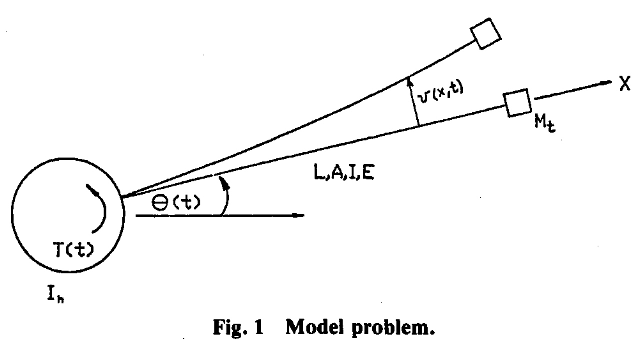

Fig. 1 Model problem.

Initial conditions:

$$\eta(\xi,0) = \phi(\xi), \qquad \dot{\eta}(\xi,0) = \psi(\xi) \tag{3}$$

with

$$\ddot{\theta} = \epsilon(\eta'')_{\xi=0} - \epsilon\delta(\eta''')_{\xi=0} + \gamma\mathfrak{Z}(\tau) \tag{4}$$

where

$$\eta = v/L, \quad \epsilon = 3I_b/I_h, \quad \lambda = c/L, \quad \delta = a/L$$

$$\xi = x/L, \quad \sigma = M_b/M_t, \quad \gamma = T_o/(I_h\alpha^2), \quad \tau = t\alpha$$

$$\zeta = I_t/(3I_b), \quad \alpha = [EI/(\rho A L^4)]^{1/2}, \quad \mathfrak{Z}(\tau) = T(t)/T_o$$

$$\phi(\xi) = \tilde{\phi}(x)/L, \quad \psi(\xi) = \tilde{\psi}(x)/(\alpha L)$$

and where the dots above variables denote differentiation with respect to τ and the primes denote differentiation with respect to ξ.

Solution to the Mathematical Model

The final form for the displacement is

$$\eta(\xi,\tau) = \bar{\eta}_F(\xi)e^{i\omega\tau} + \sum_{n=1}^{\infty} D_n V_n(\xi)e^{i\omega_n\tau} \tag{5}$$

where ω is the input frequency,

$$\bar{\eta}_F(\xi) = \frac{(\xi+\delta)}{\beta^4}[\epsilon\bar{\eta}''_F(0) - \epsilon\delta\bar{\eta}'''_F(0) + \gamma] + A\cosh\beta\xi$$

$$+ B\sinh\beta\xi + C\cos\beta\xi + D\sin\beta\xi$$

$$\bar{\eta}''_F(0) = \beta^2(A - C), \qquad \bar{\eta}'''_F(0) = \beta^3(B - D) \tag{6}$$

$$V_n(\xi) = D_{1n}\cosh\beta_n\xi + D_{2n}\sinh\beta_n\xi + D_{3n}\cos\beta_n\xi + \sin\beta_n\xi$$

$$+ [\epsilon\beta_n^2(D_{1n} - D_{3n}) - \epsilon\delta\beta_n^3(D_{2n} - 1)](\xi+\delta)/\beta_n^4 \tag{7}$$

where

$$\beta^4 = \omega^2, \qquad A_n = D_n D_{1n}$$

$$B_n = D_n D_{2n}, \qquad C_n = D_n D_{3n}$$

The D_n are constants to be determined from the initial conditions.

The angular rotation is

$$\theta(\tau) = (K_1/\beta^4)[1 - e^{i\omega\tau}] + \sum_{n=1}^{\infty}(K_2/\beta_n^4)[1 - e^{i\omega_n\tau}]$$

$$+ i\left[(K_1/\beta^2) + \sum_{n=1}^{\infty}(K_2/\beta_n^2)\right]\tau \tag{8}$$

where

$$K_1 = \gamma + \epsilon\beta^2(A - C) - \epsilon\delta\beta^3(B - D)$$

$$K_2 = D_n[\epsilon\beta_n^2(D_{1n} - D_{3n}) - \epsilon\delta\beta_n^3(D_{2n} - 1)]$$

The eigenvalues are found from the frequency equation

$$C_1[C_4(OT - SP) - C_3(NT - RP) + C_1(NS - RO)]$$

$$- C_2[-C_3(OT - SP) - C_3(MT - QP) + C_1(MS - OQ)]$$

$$+ C_4[-C_3(NT - RP) - C_4(MT - QP)$$

$$+ C_1(MR - QN)] + C_2[-C_3(NS - RO)$$

$$- C_4(MS - QO) + C_3(MR - QN)] = 0 \tag{9}$$

where

$$C_1 = 1 + (\epsilon\delta/\beta^2), \quad C_2 = -(\epsilon\delta^2/\beta), \quad C_3 = -(\epsilon/\beta^3)$$

$$C_4 = 1 - (\epsilon\delta/\beta^2), \quad M = (-1 + a_1)\cosh\beta + a_2\sinh\beta$$

$$N = (-1 + a_1)\sinh\beta + a_2\cosh\beta, \quad O = (1 + a_1)\cos\beta - a_2\sin\beta$$

$$P = (1 + a_1)\sin\beta + a_2\cos\beta, \quad Q = (1 + a_1)\sin\beta - a_3\cosh\beta$$

$$R = (-1 - a_1)\cosh\beta - a_3\sinh\beta$$

$$S = (-1 + a_1)\sin\beta - a_3\cos\beta, \quad T = (1 - a_1)\cos\beta - a_3\sin\beta$$

with

$$a_1 = \lambda\beta^2/\sigma, \qquad a_2 = \zeta\beta^3, \qquad a_3 = \beta/\sigma$$

The constants (D_{1n}, D_{2n}, and D_{3n}) in Eq. (7) satisfy

$$\begin{bmatrix} C_{1n} & C_{2n} & C_{4n} & -C_{2n} \\ -C_{3n} & C_{4n} & C_{3n} & C_{1n} \\ M_n & N_n & O_n & P_n \\ Q_n & R_n & S_n & T_n \end{bmatrix} \begin{Bmatrix} A_n \\ B_n \\ C_n \\ D_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{10}$$

where C_{1n}, C_{2n}, C_{3n}, C_{4n}, M_n, N_n, O_n, P_n, Q_n, R_n, S_n, and T_n are as before except with $\beta = \beta_n$.

The eigenfunctions $V_n(\xi,\beta_n)$ form a nonorthogonal set, and so it is not possible to match a solution to a given initial displacement and velocity distribution in the usual direct fashion. In the particular case when $\epsilon = 0$, $\zeta = 0$, and $\sigma = \infty$, they are mutually orthogonal and coincide with those of a fixed cantilever beam.

$$W_n(\xi,\mu_n) = Q_n\left[-\frac{(\sinh\mu_n + \sin\mu_n)}{(\cosh\mu_n + \cos\mu_n)}(\cosh\mu_n\xi - \cos\mu_n\xi)\right.$$

$$\left. + \sinh\mu_n\xi - \sin\mu_n\xi\right] \tag{11}$$

where Q_n are normalizing constants and $\cosh\mu_n\cos\mu_n = -1$.

Using the cantilever eigenfunctions $W_n(\xi,\mu_n)$, one makes the following expansions:

$$\bar{\eta}_F(\xi,\beta) = \sum_{j=1}^{\infty} F_j W_j(\xi,\mu_j) \tag{12}$$

$$V_n(\xi,\beta_n) = \sum_{k=1}^{\infty} G_k^{(n)} W_k(\xi,\mu_k) \tag{13}$$

$$\phi(\xi) = \sum_{m=1}^{\infty} H_m W_m(\xi,\mu_m) \tag{14}$$

The constants $D_n (n = 1,2,\ldots)$ are determined from the infinite set of linear algebraic equations

$$H_s = F_s + \sum_{n=1}^{\infty} D_n G_s^{(n)}, \qquad s = 1,\ldots,n \tag{15}$$

where $G_n^{(n)}$ and F_n are all known.

Numerical Results

The golden section method and the method of false position were both used. They yielded the nondimensional values with accuracy 10^{-11} and very close agreement with earlier results.[5,6,12] To obtain the frequency in rad/s, multiply the square of the following numbers by α. With $\lambda = 0.0225$, $\delta = 0.05$, $\epsilon = 32.0851634$, $\sigma = 0.4401257$, $\zeta = 0.00182121$, and $\alpha = 4.43064468$, the eigenvalues are 2.666914, 4.504509, 6.997543, 9.731420, 12.39665, and 15.07440. Results are presented graphically in Figs. 2 and 3, where WO_1 is the

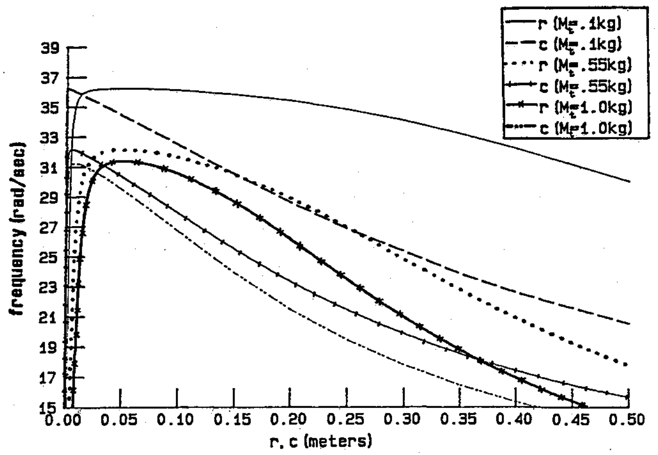

Fig. 2 First natural frequency vs r,c for the three tip mass cases.

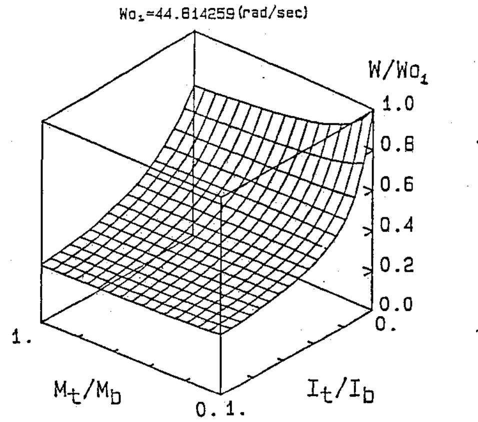

Fig. 3 First natural frequency ratio vs I_t/I_b and M_t/M_b.

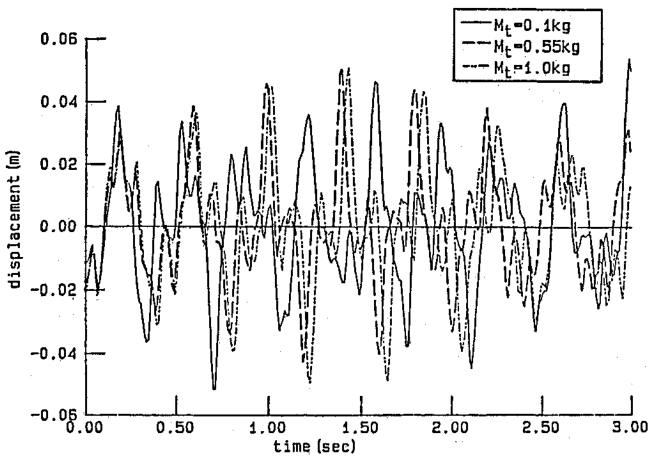

Fig. 4 Tip displacement vs time for the three tip mass cases.

natural frequency of the system with zero tip mass and W represents ω.

A dynamic parameter analysis was also performed. The input torque is an approximation to a triangular wave,

$$\Im(\tau) = \frac{8}{\pi^2}\left[\cos\frac{\pi}{P}\tau - \frac{1}{9}\cos\frac{2\pi}{P}\tau + \frac{1}{25}\cos\frac{3\pi}{P}\tau\right]$$

The tip displacement is shown in Fig. 4. The initial displacement distribution is

$$\phi(\xi) = \bar{A}\xi^2 + \bar{B}\xi^3 + \bar{C}\xi^4 + \bar{D}\xi^5$$

where

$$\bar{A} = \frac{\ddot{\theta}(0) + 6\epsilon\delta\bar{B}}{2\epsilon}, \qquad \bar{B} = \frac{(a_4/a_7)[\ddot{\theta}(0)/\epsilon] - [a_5 - (a_4a_8/a_7)]\bar{D}}{[6 - (a_4a_6/a_7)]}$$

$$\bar{C} = -\frac{a_5\bar{D} + 6\bar{B}}{a_4}, \qquad \bar{D} = -\frac{\ddot{\theta}(0)}{5}$$

with

$$a_4 = 24[1 + (1/\sigma)], \qquad a_5 = 60[1 + (2\lambda/\sigma) + (2/\sigma)]$$

$$a_6 = 6(1+\delta), \quad a_7 = 12[1 - (2\lambda/\sigma)], \quad a_8 = 20[1 - 6\zeta - (6\lambda/\sigma)]$$

The model properties are listed later for Fig. 4, which shows the tip displacement vs time for three tip mass cases: $M_t = 0.1$, 0.5547, and 1.0 kg. Initial hub angle acceleration $\ddot{\theta}(0) = 0.01$ rad/s²; torque amplitude $T_o = 1.0$ Nm, and frequency $\omega = 10.0$ Hz. System properties listed coincide with Ref. 12. The tip mass was varied, its radius always equaling half the tip mass length.

Hub:

$$\rho_h = 7750.4 \text{ kg/m}^3, \; L_h = 0.1 \text{ m}, \; a = 0.5 \text{ m}$$

Beam:

$$\rho = 2712.6 \text{ kg/m}^3, \quad E = 71 \times 10^{10} \text{ N/m}^2$$

$$L_L = 1.0 \text{ m}, \quad h = 0.003 \text{ m}, \quad b = 0.03 \text{ m}$$

Tip mass:

$$\rho_t = 7750.4 \text{ kg/m}^3, \quad L_t = 0.045 \text{ m}, \quad d = 0.045 \text{ m}$$

$M_t = 0.1$ kg:

$\lambda = 0.012711, \quad \sigma = 2.44134, \quad \zeta = 0.00010478$

$M_t = 0.5547$ kg:

$\lambda = 0.0225, \quad \sigma = 0.4401257, \quad \zeta = 0.00182121$

$M_t = 1.0$ kg:

$\lambda = 0.027384, \quad \sigma = 0.244134, \quad \zeta = 0.0048634$

with $\alpha = 4.43064468$, $\gamma = 6.69487$, $\delta = 0.05$, $\epsilon = 32.08516$ in each case.

Discussion and Conclusions

A method that can be easily numerically implemented has been presented for the analysis of a rotationally accelerated beam. The first six cantilever mode shapes and natural frequencies were used in the computations. The numerical eigenvalues obtained coincide very closely with the limiting cases, thus giving confidence that the eigenvalues obtained for the entire system are accurate. The three-dimensional plot also justifies the analysis; as the tip mass and tip inertias increase, the frequency decreases as physically expected. Finally, as Young's modulus was increased, it was seen that the forcing term for the tip displacement converged to the rigid-body case.

Acknowledgments

This material was based upon work supported by the National Science Foundation under Contract CDR 84-21415 through the Center for Robotic Systems in Microelectronics, University of California, Santa Barbara. Any opinions, findings, conclusions, or recommendations expressed in this Note are those of the authors and do not necessarily reflect the views of the foundation.

References

[1]Bauer, H. F., "Bending Vibrations of Rotating Homogeneous Satellite Booms with Time-Dependent Boundary Conditions," *Zeitschrift für Flugwissenschaften und Weltraumforschung,* Vol. 4, No. 3, 1980, pp. 121–128.

[2]Stecco, S. S., and Janigro, A., "Some Experience with the Dynamic Absorbers for Impeller Pumps," *Proceedings of the Fifth World Congress on the Theory of Machines and Mechanisms,* Vol. 1, American Society of Mechanical Engineers, New York, 1979, pp. 650–653.

[3]Hoa, S. V., "Vibrations of a Rotating Beam with Tip Mass," *Journal of Sound and Vibration,* Vol. 67, No. 3, 1979, pp. 369–381.

[4]Kojima, H., "Transient Vibrations of a Beam/Mass System Fixed to a Rotating Body," *Journal of Sound and Vibration,* Vol. 107, No. 1, 1986, pp. 149–154.

[5]Mitchell, T. P., and Bruch, J. C., Jr., "Dynamics of a Rotationally Accelerated Beam," *Journal of Guidance, Control, and Dynamics,* Vol. 12, No. 6, 1989, pp. 935–938.

[6]Mitchell, T. P., and Bruch, J. C., Jr., "Free Vibrations of a Flexible Arm Attached to a Compliant Finite Hub," *Journal of Vibrations Acoustics, Stress, and Reliability in Design,* Vol. 110, No. 1, 1988, pp. 118–120.

[7]Kane, T. R., Ryan, R. R., and Banerjee, K., "Dynamics of a Cantilever Beam Attached to a Moving Base," *Journal of Guidance, Control, and Dynamics,* Vol. 10, No. 2, 1987, pp. 139–151.

[8]Cannon, R. H., Jr., and Schmitz, E., "Initial Experiments on the End-Point Control of a Flexible One-Link Robot," *International Journal of Robotics Research,* Vol. 3, No. 3, 1984, pp. 62–75.

[9]Chassiakos, A. G., and Bekey, G. A., "Pointwise Control of a Flexible Manipulator Arm," *Symposium on Robot Control,* Barcelona, Spain, Nov. 1985, pp. 113–117.

[10]Hastings, G. G., and Book, W. J., "Experiments in the Control of a Flexible Robot Arm," *Conference Proceedings Robots 9,* June 2–6, Detroit, Michigan; also *Robotics International Journal of Society of Manufacturing Engineers,* Vols. 2:20-45 to 20-57, 1985.

[11]Sakawa, Y., Matsuno, F., and Fukushima, S., "Modeling and Feedback Control of a Flexible Arm," *Journal of Robotic Systems,* Vol. 2, No. 4, 1985, pp. 453–472.

[12]Rakhsha, F., and Goldenberg, A. A., "Dynamics Modeling of a Single-Link Flexible Robot," *Proceedings of the IEEE International Conference on Robotics and Automation,* 1985, pp. 984–989.

[13]Wie, B., and Bryson, A. E., "Pole-Zero Modeling of Flexible Space Structures," *Journal of Guidance, Control, and Dynamics,* Vol. 11, No. 6, 1988, pp. 554–561.

Guidance for Asteroid Rendezvous

M. Guelman*

RAFAEL, Ministry of Defense, Haifa, Israel

Introduction

THE use of electric propulsion at very low thrust levels for different missions to asteroids and comets is a very attractive option. Various aspects of the guidance and navigation systems for these missions were studied previously.[1-4]

The rendezvous with a celestial object has a number of phases: 1) cruise or transfer to the vicinity of the celestial body, 2) approach, and 3) maneuvers near the celestial body. This Note is concerned with the approach phase.

In Ref. 4, an onboard autonomous guidance system was proposed based on a steerable platform carrying an imaging system that points at the celestial body measuring the line-of-sight (LOS) inertial rate and a laser system that is aligned with the imaging system and measures the relative range to the celestial body.

For a spacecraft with an autonomous onboard tracking system, the relative velocity components along and normal to the LOS as well as the relative range are the most adequate inputs for a guidance law. The use of LOS coordinates for the guidance law was considered previously in Ref. 5, where the commanded acceleration along the LOS was defined as a function of the square root range, and for the accelerations normal to the LOS, two different cases were considered: 1) proportional to the LOS rate as in proportional navigation[6] or 2) proportional to the relative velocity normal to the LOS.

In this work, a guidance law in LOS coordinates, which is a generalization of the exponential type of guidance studied in Ref. 7, is considered for the rendezvous with an asteroid.

Received Oct. 18, 1988; revision received April 2, 1990.

*Deputy Director, Guidance Division, P.O. Box 2250.

System Definition

Figure 1 depicts the in-plane rendezvous geometry, where S is an active spacecraft able to apply a continuous thrust controlled both in amplitude and direction, and A is a passive object, a celestial body of small dimensions.

Spacecraft rendezvous maneuvers in the vicinity of the celestial object will be considered. The duration of these maneuvers relative to the celestial body orbital period being negligible, a constant rectilinear motion for the celestial body will be assumed.

Using an inertial system of coordinates centered on A, the relative equations of motion are defined by

$$R\ddot{\theta} + 2\dot{R}\dot{\theta} = a_\theta \tag{1}$$

$$\ddot{R} - R\dot{\theta}^2 = a_r \tag{2}$$

where R, $\dot{R}$, and $\ddot{R}$ are the relative SA range, range rate, and range acceleration, respectively; $\dot{\theta}$ and $\ddot{\theta}$ are the LOS angular rate and acceleration; and a_r and a_θ are the acceleration components along and normal to the LOS acting on the S vehicle.

The use of autonomous guidance systems for interception is widely known. One of the simplest—and probably the most successful—of the guidance laws for interception is proportional navigation, where the intercepting vehicle maneuvers normal to the LOS are made proportional to its angular rate. This assures that the LOS rate approaches zero for nonmaneuvering targets, a collision course is achieved, and a successful interception is ensured.

While interception is the matching of the position vectors of two vehicles, the rendezvous of two vehicles is the simultaneous matching of both their position and velocity vectors. The use of the LOS rate, or possibly the relative velocity normal to the LOS, is only part of the rendezvous problem, since the relative velocity must be zero when the two vehicles meet. The active vehicle acceleration then must include a component along the LOS that reduces the relative closing velocity to zero as the two vehicles approach one another.

A guidance rendezvous law of the form

$$a_{\theta c} = c_1 R\dot{\theta} \tag{3}$$

$$a_{rc} = c_2\dot{R} - c_3 R \tag{4}$$

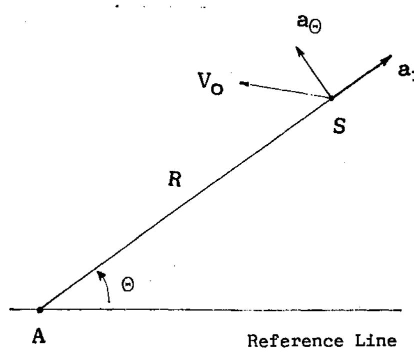

Fig. 1 Relative geometry.